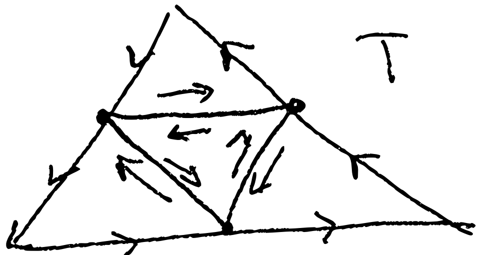
Lecture 24 (11/19/21).

We now circle back and show that it suffices to assume that $f: G \to \mathbb{C}$ is $\mathbb{C}$-diff. at every point to conclude it is analytic (which requires f' to also be cont.). Recall the cont. of f' was used to be able to apply Leibniz rule in the pf of Baby-CIF, which gave the local power series expansion of f ($\Rightarrow C^\infty$).

<u>Goursat's Thm</u>. Assume $f: G \to \mathbb{C}$ is $\mathbb{C}$-diff. in G. Then f is analytic.

<u>Pf</u>. Analyticity is local property, so may assume G is a disk $B(a,r)$. To conclude analyticity, we use Morera, which applies since $\mathbb{C}$-diff. $\Rightarrow$ continuity of f. Thus, we must show that $\int_T f\,dz = 0$ for every triangular path in $B(a,r)$.



Split T into 4 triang. paths T_1, T_2, T_3, T_4 by midpoints as in pic. We have

$$\int_T f\,dz = \sum_{j=1}^{4} \int_{T_j} f\,dz .$$

Let $T^{(1)}$ denote a triang. path T_j s.t.

$$\left|\int_{T^{(1)}} f\,dz\right| \geq \max_{j\in\{1,\dots,4\}} \left|\int_{T_j} f\,dz\right| . \Rightarrow$$

$$\left|\int_T f\,dz\right| \leq 4\left|\int_{T^{(1)}} f\,dz\right| . \quad (*)$$

Continuing in this way to split $T^{(1)}$ into 4 smaller triang. paths and letting $T^{(2)}$ be one of them w/ the same maximal property, we obtain a sequence of triang. paths $T^{(1)}, T^{(2)}, \dots$

Such that the closed triangles (i.e. the insides + triang. path) $\Delta^{(1)}, \Delta^{(2)}, \ldots$ form a decreasing sequence

$$\Delta \supseteq \Delta^{(1)} \supseteq \Delta^{(2)} \supseteq \ldots$$

Since the diam $\Delta^{(k)} \leq 2^{-k} \operatorname{diam} \Delta \to 0$

Caratheodory $\Rightarrow \bigcap_{k=1}^{\infty} \Delta^{(k)} = \{z_0\}$.

Since f is $\mathbb{C}$-diff. at z_0, if we pick $\varepsilon > 0$ $\exists$ $\delta > 0$ s.t.

$$|f(z) - f(z_0) - f'(z_0)(z - z_0)| < \varepsilon |z - z_0|$$

for $|z - z_0| < \delta$. By induction, using (*) and similar for $T^{(k)}$, we get

$$\left|\int_T f\,dz\right| \leq 4^k \left|\int_{T^{(k)}} f\,dz\right| = \left\{\begin{matrix} f(z_0) \,\& \\ f'(z_0)(z - z_0) \\ \text{anal. in} \\ z \end{matrix}\right\}$$

$$4^k \left|\int_{T^{(k)}} \left(f(z) - f(z_0) - f'(z_0)(z - z_0)\right) dz\right|$$

$$\leq 4^k \cdot \varepsilon \cdot \operatorname{diam} \Delta^{(k)} \quad \ell(T^{(k)})$$

length

$\leq 4^k \cdot 2^{-k} \operatorname{diam} \Delta \cdot 2^{-k} \ell(T) \cdot \varepsilon = \varepsilon \operatorname{diam} \Delta \, \ell(T)$

Since $\varepsilon > 0$ arbitrary, $\int_T f \, dz = 0$ and hence f is analytic by Morera. $\boxtimes$